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Stability of Spatially and Temporally Sampled Distributed Parameter Feedback Systems

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STABILITY OF SPATIALLY AND TEMPORALLY SAMPLED DISTRIBUTED PARAMETER FEEDBACK SYSTEMS

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Summary

Feedback for the control of distributed parameter systems that incorporate spatial and temporal sampling is studied. A method that uses a modal approach is developed for the stability analysis of such systems. To illustrate application of this method, the stability of a simple distributed parameter electromechanical system is examined. The electromechanical system analyzed is one that has been used to model electric field control of Rayleigh-Taylor instability at fluid-fluid interfaces.

I. Introduction

Questions concerning the control and stability of continuum or distributed parameter systems appear in such diverse areas as aerodynamics, plasma physics, electrohydrodynamics, structural mechanics and the technologies of nuclear reactors, particle accelerators and chemical reactors.

In boundary layer flow (1), thermonuclear machines (2), magnetohydrodynamic channel flows (3), and electric field levitation (4), stability is a primary consideration. Feedback methods are used or have been proposed to control particle beams in accelerators (5), spatial oscillations in nuclear reactors (6), fluids in normally unstable configurations (7) and processes in chemical reactors (8,9,10). Although most of the reports about distributed parameter control systems have been in the context of specific problems, a general theory of distributed parameter control systems is developing (11, 12, 13, 14, 15, 16, 17).

The ideal way to control a distributed parameter system would be to measure a variable describing the state of the system at each point, and feed back to the point a function of this describing variable as a controlling force. Generally, one is limited to spatially sampling the describing variable, and applying the controlling force over a finite region of the continuum. It is an intuitive notion that, as the number of sampling stations is increased, better control can be exerted over the continuum. For a system with a large number of sampling stations, it is not practical to make the feedback circuitry for each station independent of that for the other stations. A type of scanning or time-sharing scheme capable of reducing the hardware requirements is desirable; such a time-sharing system would scan the continuum, examining each sampling point every T seconds.

This paper presents a method for stability analysis of distributed parameter feedback systems that incorporate both spatial and temporal sampling. This method is applicable to systems whose open loop dynamics can be described in terms of spatial modes.

The distributed parameter feedback problem to be analyzed is described in Section II. In the third section, the partial differential equation describing the closed system is manipulated into a first order difference equation. The stability of the system is then determined by examining the stability of the difference equation. To illustrate the use of the method developed in Section III, the stability of a simple electromechanical system is analyzed in Section IV. The system analyzed is one that has been used to model electric field control of Rayleigh-Taylor instability (3) at fluid-fluid interfaces (7).

II. Spatially and Temporally Sampled Feedback

Consider a continuum divided into S regions (see Fig. 1) and a variable $\underline{\xi}(\underline{r},t)$, associated with the continuum, that is described by the partial differential equation:

$$\frac{\partial \underline{\xi}}{\partial t} = \underline{L}\,\underline{\xi} + \underline{f} \tag{1}$$

The matrix $\underline{L}$ is n x n and is the sum of $\underline{A}$, a constant matrix, and $\underline{D}$, a matrix whose components are spatial differential operators. The feedback force $\underline{f}$ that controls the variable $\underline{\xi}$ over the continuum is spatially and temporally discrete.

For linear feedback, the feedback force to the i^{th} region, R_i, can be written:

$$\underline{f} = \underline{M}\,\underline{\xi}(\underline{r}_i, t_{ij}) \quad ; \quad \text{for } t_j \leq t \leq t_{j+1}$$

where $\underline{M}$ is the feedback gain matrix, $\underline{r}_i$ is the position* within R_i at which $\underline{\xi}$ is measured, t_{ij} is the time at which $\underline{\xi}$ was measured and $(t_{j+1} - t_j)$ is the length of time over which the force remains unchanged.

An expression for the feedback term for uniform temporal sampling** of an S-region system can be obtained with the aid of Tables 1 and 2. At time $(\beta S + k)\tau$ the $(k+1)^{st}$ region is sampled, and the controlling force at the $(k+1)^{st}$ region changes from $\underline{M}\,\underline{\xi}(\underline{r}_{k+1}, (\beta S - S + k)\tau)$ to $\underline{M}\,\underline{\xi}(\underline{r}_{k+1}, (\beta S + k)\tau)$ while the controlling forces applied to the other regions remain unchanged. The force that is applied to the continuum during the interval, $(\beta S + k)\tau \leq t \leq (\beta S + k + 1)\tau$ is tabulated in Table 2; $\underline{f}$ for region R_i can be written:

$$\underline{f} = \begin{cases} \underline{M}\,\underline{\xi}(\underline{r}_i, (\beta S + i - 1)\tau) \ ; & 1 \leq i \leq k+1 \\ \underline{M}\,\underline{\xi}(\underline{r}_i, (\beta S - S + i - 1)\tau) \ ; & k+2 \leq i \leq S \end{cases} \qquad (2)$$

$$\text{for} \quad (\beta S + k)\tau \leq t \leq (\beta S + k + 1)\tau$$

* Schemes that feed back to R_i a weighted average of ξ over R_i can be analyzed with the method that is developed. Single point feedback is chosen for illustrative simplicity.

** The method of stability analysis that is developed does not require uniform temporal sampling. Uniform temporal sampling is chosen for illustrative simplicity.

Time Interval $t_1 \leq t \leq t_2$			REGION						
t_1	t_2		R_1	R_2		R_k	R_{k+1}		R_S
0	$S\tau$	VARIABLE MEASURED	$\underline{\xi}(\underline{r}_1, 0)$	$\underline{\xi}(\underline{r}_2, \tau)$		$\underline{\xi}(\underline{r}_k, (k-1)\tau)$	$\underline{\xi}(\underline{r}_{k+1}, k\tau)$		$\underline{\xi}(\underline{r}_S, (S-1)\tau)$
$S\tau$	$2S\tau$		$\underline{\xi}(\underline{r}_1, S\tau)$	$\underline{\xi}(\underline{r}_2, (S+1)\tau)$		$\underline{\xi}(\underline{r}_k, (S+k-1)\tau)$	$\underline{\xi}(\underline{r}_{k+1}(S+k)\tau)$		$\underline{\xi}(\underline{r}_S, (2S-1)\tau)$
					. . .			. . .	
$\beta S\tau$	$(\beta+1)S\tau$		$\underline{\xi}(\underline{r}_1, \beta S\tau)$	$\underline{\xi}(\underline{r}_2, (\beta S+1)\tau)$		$\underline{\xi}(\underline{r}_k, (\beta S+k-1)\tau)$	$\underline{\xi}(\underline{r}_{k+1}, (\beta S+k)\tau)$		$\underline{\xi}(\underline{r}_S, (\beta S+S-1)\tau)$

$\beta = 0,1,2 \ldots\ldots$; $r = 0,1,2, \ldots S-1$

TABLE 1

Entries are the variables measured by the scanning system

Region	Feedback $(\underline{f})$
R_1	$\underline{M}\ \underline{\xi}(\underline{r}_1, \beta S\tau)$
R_2	$\underline{M}\ \underline{\xi}(\underline{r}_2, (\beta S + 1)\tau)$
.	
.	
.	
R_r	$\underline{M}\ \underline{\xi}(\underline{r}_k, (\beta S + k - 1)\tau)$
R_{r+1}	$\underline{M}\ \underline{\xi}(\underline{r}_{k+1}, (\beta S + k)\tau)$
R_{r+2}	$\underline{M}\ \underline{\xi}(\underline{r}_{k+2}, (\beta S - S + k + 1)\tau)$
.	
.	
.	
R_S	$\underline{M}\ \underline{\xi}(\underline{r}_S, (\beta S - 1)\tau)$

TABLE 2

The entry for region R_i is the feedback force applied to region R_i during the time interval $(\beta S + k)\tau \leq t \leq (\beta S + k + 1)\tau$

We wish to develop a method for determining if the system described by equations (1) and (2) is asymptotically stable $(\lim_{t\to\infty} \underline{\xi}(\underline{r},t) = \underline{0})$.

The class of systems considered is restricted to those for which the open loop solution $(\underline{f} = 0)$ is of the form:

$$\underline{\xi}(\underline{r},t) = \sum_{m=1}^{\infty} \underline{z}_m(t)\, \Phi_m(\underline{r})$$

The problem is further restricted by including only those systems for which the spatial modes $\left\{\Phi_m(\underline{r})\right\}$ are orthonormal in some sense over the continuum.* Under these restrictions, the open loop equation is equivalent to a set of pairs of equations:

$$\underline{D}\Phi_m(\underline{r}) = \Phi_m(\underline{r})\,\underline{E}_m \tag{3a}$$

$$\dot{\underline{z}}_m = (\underline{A} + \underline{E}_m)\,\underline{z}_m \tag{3b}$$

$$m = 1, 2, 3, \ldots .$$

III. A Method of Stability Analysis

The orthononormality of the $\left\{\Phi_m(\underline{r})\right\}$ makes it possible to write $\underline{f}$ as:

$$\underline{f} = \sum_{m=1}^{\infty} \underline{f}_m \Phi_m(\underline{r})$$

where $\underline{f}_m = \sum_{i=1}^{S} a_1(m)\underline{M}\,\underline{\xi}(\underline{r}_i, t_{ik})$ and (for systems whose spatial modes satisfy Sturm-Liouville equations):

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* For the case when $\underline{\xi}$ is two-dimensional, the highest spatial derivative in $\underline{D}$ is second order and the equation $\underline{D}\Phi = \underline{E}\Phi$ is separable in some coordinate system; the functions $\Phi_m(x_1)$, $\Phi_m(x_2)$, $\Phi_m(x_3)$ into which $\Phi_m(\underline{r})$ is separable satisfy Sturm-Liouville equations and the orthonormality condition:

$$\int_{\text{boundary a}}^{\text{boundary b}} \Phi_m(x_i)\,\Phi_n(x_i)q(x_i)dx_i = \delta_{mn}$$

Other systems may satisfy other orthonormality conditions.

$$a_i(m) = \int_{R_i} \Phi_m(\underline{r})q(\underline{r})d\underline{r}$$

The orthonormal properties of the $\left\{\Phi_m(\underline{r})\right\}$ are again used to write the solution to the closed loop equation as:

$$\underline{\xi}(\underline{r},t) = \sum_{m=1}^{\infty} \underline{w}_m(t)\Phi_m(\underline{r})$$

Equations (1) and (3.a) require that:

$$\dot{\underline{w}}_m = (\underline{A} + \underline{E}_m)\underline{w}_m + \underline{f}_m \quad ; \quad m = 1,2,3 \ldots. \qquad (4)$$

Note that, while the spatial dependence is implicit in equation (4), the explicit spatial nature of the problem has vanished. Thus, the problem is now essentially one of an infinite number of coupled, lumped modes. On physical grounds it is apparent that an accurate description of the dynamics of the system can be obtained by considering a finite number, of these modes, and by approximating $\underline{\xi}(\underline{r},t)$ as:

$$\underline{\xi}(\underline{r},t) \cong \sum_{m=1}^{\alpha} \underline{w}_m(t)\Phi_m(\underline{r})$$

It is convenient at this point to define a state vector $\underline{x}$:

$$\underline{x} = \begin{pmatrix} \underline{w}_1 \\ \vdots \\ \underline{w}_\alpha \end{pmatrix}$$

$\underline{x}$ satisfies the equation:

$$\dot{\underline{x}} = \underline{B}\,\underline{x} + \underline{g} \quad ; \qquad (\beta S + k)\tau \leq t \leq (\beta S + k + 1)\tau$$

where:

$$\underline{B} = \begin{pmatrix} \underline{A}+\underline{E}_1 & \underline{0} & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{A}+\underline{E}_2 & \underline{0} & \cdots & \underline{0} \\ \vdots & & & & \vdots \\ \underline{0} & & & & \underline{A}+\underline{E}_\alpha \end{pmatrix} \quad \text{and} \quad \underline{g} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_\alpha \end{pmatrix}$$

Using the result obtained in the appendix, that $\underline{g} = \sum_{i=1}^{S} \underline{G}_i \, \underline{x}(t_{i\beta})$,

where the $\underline{G}_i$ are the functions of the $a_i(m)$, $\Phi_m(\underline{r}_i)$ and $\underline{M}$ and

where $t_{i\beta} = \begin{cases} [\beta S + (i-1)]\tau & ; \quad 1 \leq i \leq k+1 \\ [\beta S + S + (i-1)]\tau & ; \quad k+2 \leq i \leq S \end{cases}$,

the state equation may be written:

$$\dot{\underline{x}} = \underline{B}\,\underline{x} + \sum_{i=1}^{S} \underline{G}_i \, \underline{x}\,(t_{i\beta}) \;\; ; \;\; (\beta S + k)\tau \leq t \leq (\beta S + k + 1)\tau \tag{5}$$

The solution to equation (5) valid for $(\beta S + k)\tau \leq t \leq (\beta S + k + 1)\tau$ is:

$$\underline{x}\,(t) = e^{\underline{B}[\,t - (\beta S + k)\tau\,]}\underline{x}[\,(\beta S + k)\tau\,] + \sum_{i=1}^{S} [e^{\underline{B}[\,t - (\beta S + k)\tau\,]} - I\,]\underline{B}^{-1}\underline{G}_i \, \underline{x}\,(t_{i\beta})$$

At $t = (\beta S + k + 1)\tau$:

$$\underline{x}[\,(\beta S + k + 1)\tau\,] = e^{\underline{B}\tau}\,\underline{x}\,[\,(\beta S + k)\tau\,] + \sum_{i=1}^{S} \underline{P}_i \, \underline{x}\,(t_{i\beta}) \tag{6}$$

where $\underline{P}_i = [e^{\underline{B}\tau} - I]\,\underline{B}^{-1}\underline{G}_i$. Difference equation (6) relates the state of the system at time $(\beta S + k + 1)\tau$ to the states of the system at $(\beta S + k)\tau$, $(\beta S + k - 1)\tau$, ... $(\beta S + k - S + 1)\tau$.

To determine system stability, we use standard techniques for analyzing difference equations.* In the appendix, the S equations represented by equation (6) are combined to obtain the single equation:

$$\underline{x}[(m+S)\tau] = \underline{Q}_1\, \underline{x}(m\tau) + \underline{Q}_2\, \underline{x}[(m-1)\tau] + \underline{Q}_3\, \underline{x}[(m-2)\tau] + \ldots$$
$$\underline{Q}_{S-1}\, \underline{x}[(m+2-S)\tau] + \underline{Q}_S\, \underline{x}[(m+1-S)\tau] \tag{7}$$

where $m = \beta S$. This $(2S - 1)$ order difference equation may be reduced to the first order equation:

$$\underline{X}[(m+1)\tau] = \underline{Q}\, \underline{X}(m\tau) \tag{8}$$

where

$$\underline{X}(m\tau) = \begin{matrix} \underline{x}_1(m\tau) \\ \vdots \\ \underline{x}_S(m\tau) \end{matrix} \quad ; \quad \underline{x}_i(k\tau) = \underline{x}(k\tau + (1-i)\tau)$$

and

$$\underline{Q} = \begin{matrix} \underline{0} & \underline{I} & \underline{0} & \cdot & \cdot & \cdot & \underline{0} \\ \underline{0} & \underline{0} & \underline{I} & \cdot & \cdot & \cdot & \underline{0} \\ \vdots & & & & & & \vdots \\ \underline{0} & \underline{0} & \underline{0} & \cdot & \cdot & \cdot & \underline{I} \\ \underline{Q}_1 & \underline{Q}_2 & \underline{Q}_3 & \cdot & \cdot & \cdot & \underline{Q}_S \end{matrix}$$

Equation (8) is asymptotically stable when the magnitude of the eigenvalues of $\underline{Q}$ are less than one $(|\lambda(\underline{Q})| < 1)$. The positions of the eigenvalues of the $(n\,S) \times (n\,S)$ matrix $\underline{Q}$ determine the stability of the S- station nth order distributed parameter system.

It is evident from equation (5) that, in contrast to continuous time feedback, temporally sampled feedback leaves the open loop

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* See for example: David P. Lindorff, Theory of Sampled-Data Control Systems, Wiley, 1965

natural frequencies of the system unchanged. Whereas continuous time feedback stabilizes by forcing the natural frequencies into the left half plane (s- plane), discrete time feedback stabilizes by forcing the "discrete natural frequencies" (the eigenvalues of difference equation (8)) into the unit circle (of the λ plane).

IV. An Example

The stability of the feedback system shown in figure (2.a) will be analyzed to demonstrate application of the method and to illustrate some effects of temporal sampling. This situation is of interest, for it is one that is used to model the system indicated schematically in figure (2.b) of a conducting liquid supported against gravity by hydrostatic pressure[7, 18, 19]. With no feedback, the system in figure (2.b) is unstable; $v(\underline{r},t)$ represents electric field feedback used to stabilize the Rayleigh-Taylor instability at the liquid-gas interface. The instability that results in figure (2.a) when V_o exceeds $V_{max} = \sqrt{\frac{dT}{2\varepsilon_o}}\frac{\pi d}{L}$ is mathematically similar but less complex than the Rayleigh-Taylor interface instability. The use of $v(y,t)$ to stabilize the membrane system for values of V_o greater than V_{max} corresponds in a rough way to the use of electric field feedback to stabilize the liquid-gas interface in figure (2.b).

The linearized equation of motion for transverse displacements $\xi(y,t)$, of the conducting membrane is:

$$\rho \frac{\partial^2 \xi}{\partial t^2} = T\frac{\partial^2 \xi}{\partial y^2} - \beta\frac{\partial \xi}{\partial t} + \frac{2\varepsilon_o V_o^2}{d^3}\xi - \frac{2\varepsilon_o V_o}{d^2} v \qquad (9)$$

where ρ is the mass per unit area, β is the effective damping coefficient, and T is the tension per unit depth. For purposes of illustration and ease of computation, S is taken to be one, y_1 to be L/2 and only the displacement, $\xi(y_1,k\tau)$, is fed back: $v(y,t) = A\xi(^L/2,k\tau)$; $k\tau \leq t \leq (k+1)\tau$. Equation (9) may be rewritten in the form of equation (1):

$$\frac{\partial}{\partial t}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ \theta\frac{\partial^2}{\partial y^2}+N & -\Delta\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} + \begin{pmatrix}0 & 0\\ -M & 0\end{pmatrix}\begin{pmatrix}\xi_1(^L/2,k\tau)\\ \xi_2(^L/2,k\tau)\end{pmatrix} \tag{10}$$

where $\xi_1 = \xi$, $\xi_2 = \frac{\partial\xi}{\partial t}$, $\theta = T/\rho$, $\Delta = \beta/\rho$, $N = \frac{2\varepsilon_o V_o^2}{\rho d^3}$ and $M = \frac{2\varepsilon_o V_o}{\rho d^2}$.

We wish to find the region in M-N-τ-Δ space in which the membrane is stable; if the feedback is effective, the system will be stable for $N > N_{max}$ ($N_{max} = N(V_{max})$) for some values of M,Δ,τ.

Melcher[1] has examined spatially sampled feedback applied to lossless membranes ($\Delta = 0$, $\tau = 0$); figure (3) is a summary of his results for S = 1,2,3,4 and $S \to \infty$. These results indicate that the region of stability in the N-M plane enlarges while retaining approximately the same shape as S increases. While we consider only a single station, on the basis of Melcher's results we can conjecture that, with a temporally sampled system, the region of stability enlarges while retaining the same basic shape as the number of stations is increased.

A typical spatially sampled system (S = 4) is stable to the left of line 1 (refer to figure (3)), to the left of line 2, and below line 3. The instability that occurs to the right of line 1 is a static (purely exponential) instability that results because

The feedback gain M is not large enough to counteract the unstabilizing effect of V_o. The feedback system is unable to control the spatial modes whose nodes occur at the sampling points; the static instability to the right of line 2 results when one of the unmonitored modes becomes unstable. Above line 3, overstability (instability characterized by growing oscillations) occurs as a result of overly compensating for the unstabilizing effect of V_o. Systems that feed back a spatial average (over R_i) rather than a single point value of ξ do not exhibit this overstability; averaging schemes do, however, exhibit instability to the right of line 2.

It follows from equation (10) that:

$$\underline{A} = \begin{pmatrix} 0 & 1 \\ N & -\Delta \end{pmatrix} \; ; \; \underline{D} = \begin{pmatrix} 0 & 0 \\ \theta \frac{\partial^2}{\partial y^2} & 0 \end{pmatrix} \quad \text{and } \underline{M} = \begin{pmatrix} 0 & 0 \\ -M & 0 \end{pmatrix}$$

For this system, the $\left\{\Phi_m(y)\right\}$,

$$\Phi_m(y) = \sqrt{\frac{2}{L}} \sin \frac{m\pi y}{L} = \sqrt{\frac{2}{L}} \sin k_m y$$

are clearly orthonormal and complete over (0,L)

The eigenmatrix $\underline{E}_m$ is:

$$\underline{E}_m = \begin{pmatrix} 0 & 0 \\ -\theta k_m^2 & 0 \end{pmatrix}$$

and $\underline{f}_m$ is:

$$\underline{f}_m = \begin{cases} \begin{pmatrix} 0 \\ \dfrac{2\sqrt{2}\, LM\xi(L/2, k\tau)}{m\pi} \end{pmatrix} & ; \text{ m odd.} \\ \underline{0} & ; \text{ m even.} \end{cases}$$

Melcher[1] argues that α must be at least 2S + 1 to obtain an accurate description of the dynamics of the system. For $\alpha = 3$, the equation for the state vector $\underline{x}$, is:

$$\underline{x} = \begin{pmatrix} \underline{A}+\underline{E}_1 & \underline{0} & \underline{0} \\ \underline{0} & \underline{A}+\underline{E}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{A}+\underline{E}_3 \end{pmatrix} \underline{x} + \frac{2\sqrt{2}LM}{\pi} \begin{pmatrix} 0 \\ \xi(L/2,k\tau) \\ 0 \\ 0 \\ 0 \\ \frac{\xi(L/2,k\tau)}{3} \end{pmatrix}$$

or: $\underline{x} = \underline{B}\,\underline{x} + \underline{G}_1\underline{x}\,(k\tau)$.

Where $\underline{G}_1$ is the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-4M}{\pi}\sin k_1y_1 & 0 & \frac{-4M}{\pi}\sin k_2y_1 & 0 & \frac{-4M}{\pi}\sin k_3y_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-4M}{3\pi}\sin k_1y_1 & 0 & \frac{-4M}{3\pi}\sin k_2y_1 & 0 & \frac{-4M}{3\pi}\sin k_3y_1 & 0 \end{pmatrix}$$

The state vector satisfies the difference equation:

$$\underline{x}\,[(k+1)\tau] = e^{\underline{B}\tau}\,\underline{x}(k\tau) + (e^{\underline{B}\tau} - \underline{I})\underline{B}^{-1}\,\underline{G}_1\,\underline{x}\,(k\tau)$$

and so $\underline{Q}$ is the 6 x 6 matrix:

$$e^{\underline{B}\tau} + (e^{\underline{B}\tau} - \underline{I})\underline{B}^{-1}\,\underline{G}_1$$

The eigenvalue equation, $\det(\underline{I}\lambda - \underline{Q}) = 0$, factors into a second order equation and a fourth order equation. Setting $\lambda = \frac{1+s}{1-s}$ transforms the requirement that $|\lambda| < 1$ into the requirement that $\mathrm{Re}(s) < 0$. Application of the Routh-Hurwitz conditions to insure $\mathrm{Re}(s) < 0$ results in the following conditions (valid for small τ)* that must be satisfied for the system to be stable:

(a) $\Delta' > 0$

(b) $N' < 4\pi^2$

(c) $\Delta' - 2\left(\Delta'^2 + \frac{2M'}{3\pi}\right)\tau/T_{n=1} > 0$

(d) $N'^2 - \left(10\pi^2 + \frac{8M'}{3\pi}\right)N' + \frac{104M'\pi}{3} + 9\pi^2 + \frac{16M'\Delta'}{3\pi}(N' - 13\pi^2)\tau/T_{n=1} > 0$

(e) $$\frac{4M'}{3\pi} + 5\pi^2 - N' + \Delta'^2 + \frac{4}{\Delta'}\Big(N'\Delta'^2 - 5\Delta'^2\pi^2 - \Delta'^4 + \frac{7M'\Delta'^2}{3\pi} + \frac{M'N'}{3\pi} + M'\pi - \frac{8M'^2}{9\pi^2}\Big)\tau/T_{n=1} > 0$$

(f) $$M'^2 - \frac{3\pi}{4}(16\pi^2 - \Delta'^2)M' - \frac{9\pi^2}{16}(N'\Delta'^2 - 16\pi^4 - 5\Delta'^2\pi^2) + 2\Big\{\frac{-8}{3\pi\Delta'}M'^3 + \Big(\frac{32\pi^2 - 9\Delta'^2}{\Delta'}\Big)M'^2 + \Big(3N'\Delta'\pi - 51\Delta'\pi^3 - 24\frac{\pi^5}{\Delta'} - \frac{9\Delta'^3\pi}{2}\Big)M' + \frac{27N'\Delta'^3\pi^2}{8} + \frac{45N'\Delta'\pi^4}{4} - \frac{9N'^2\Delta'\pi}{8} - \frac{477\pi^6\Delta'}{8} - \frac{135\pi^4\Delta'^3}{8}\Big\}\tau/T_{n=1} > 0$$

where $\Delta' = \sqrt{\frac{\rho}{TL^2}}\,\Delta$; $M' = \frac{\rho}{TL^2}M$; $N' = \frac{\rho}{TL^2}N$ and $T_{n=1} = 2\sqrt{\frac{\rho L^2}{2}}$ = period of the $n = 1$ mode with $V_o = 0$ and $v(y,t) = 0$.

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* $\tau \ll \begin{cases} \frac{1}{\Delta} \\ (\theta k_1^2 - N - \Delta^2/4)^{-\frac{1}{2}} \end{cases}$

These inequalities define a region in M'-N'-Δ'-τ space (or M'-N' space with Δ' and τ as parameters) in which the membrane system is stable. The stable regions for a lightly damped system (a liquid like water) and for a heavily damped system (a liquid like glycerine or molten glass) are plotted in figure (4.a) and figure (4.b) for various sampling rates.

From figure (4) it is evident that the major effect of temporal sampling on this system is to make it more susceptible to overstability. Additional analysis shows that a lossless system is always overstable to temporal sampling. Stability of a system incorporating temporal sampling is made possible only by the presence of loss. This sensitivity to overstability is not surprising, for if τ is too large, there is an effective lag in the feedback loop that results in a net addition of energy to the membrane for each membrane oscillation, rather than a net removal of energy for each oscillation.

The price paid for the introduction of temporal sampling, in order to reduce the hardware requirements of a distributed parameter feedback system, is the increased susceptibility of the system to overstability.

V. Conclusion

Application of the method developed for the analysis of spatially and temporally sampled feedback is conceptually straightforward: to determine stability, examine the locations of the eigenvalues of a matrix. However, for even the simplest cases, ($n = 2$), this matrix must be at least $(4S^2 + 2S) \times (4S^2 + 2S)$ to reresent the system accurately. Therefore, while application of the method to an S- station system is straightforward, S neet not be very large before the amount of computation required makes implementation of the method unattractive. Although a complete analysis of an S- station system, resulting in definition of the stable region in terms of the feedback gain, sampling rate, etc., might be unmanageable, the following type of partial analysis will yield parameter values that insure stability:

(1) Completely analyze the system for the case of S = 1 or S = 2, and choose parameter values (M,N,τ ...) that insure stability;

(2) Use these parameter values to examine the positions of the eigenvalues for the S- station system;

(3)
a) If the S- station system is found to be stable but the parameter values are not as desired (perhaps N is too small), increment the values and repeat (2).

b) If the S- station system is unstable, change the parameter values slightly and repeat (2).

Melcher's results[7] indicate that the major change in stability properties as S increases is a change in the scales of the M and N axes. This suggests scaling the parameter values found in (1) before advancing to step (2). The details of this scaling behavior are under investigation.

To a certain extent, the increasing complexity of the stability analysis with increasing S is due to the inclusion of the effects of spatial boundaries. For many fluid systems, the boundaries have negligible effect on the system dynamics. For such continuum systems, Fourier transform or wavetrain techniques have been successfully applied to the analysis of feedback systems incorporating large numbers of spatial samples[7,19]; such techniques may also prove useful for reducing the complexity of analysis of systems incorporating temporal sampling in conjunction with a large number of spatial samples.

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<u>Appendix</u>

1. Evaluation of $\underline{g}$:

$$g = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{f}_\alpha \end{pmatrix} = \sum_{i=1}^{S} \begin{pmatrix} a_i(1) \;\; \underline{M}\,\underline{\xi}(\underline{r}_i, t_{i\beta}) \\ a_i(2) \;\; \underline{M}\,\underline{\xi}\,(\underline{r}_i, t_{i\beta}) \\ \cdot \\ \cdot \\ \cdot \\ a_i(\alpha) \;\; \underline{M}\,\underline{\xi}(\underline{r}_i, t_{i\beta}) \end{pmatrix}$$

Consider the following four $(n\alpha)x(n\alpha)$ matrices:

$$\underline{A}_{id} = \underbrace{\begin{pmatrix} a_i(1) & & & & & & & \\ & \ddots & & & & & & \\ & & a_i(1) & & & & \underline{0} & \\ & & & a_i(2) & & & & \\ & \underline{0} & & & \ddots & & & \\ & & & & & a_i(2) & & \\ & & & & & & \ddots & \\ & & & & & & a_i(\alpha) & \\ & & & & & & & \ddots \;\; a_i(\alpha) \end{pmatrix}}_{n\alpha} \; n\alpha$$

(blocks of size n)

$$\underline{M}_d = \underbrace{\begin{pmatrix} \underline{M} & \underline{0} & \cdot & \cdot & \cdot & \cdot & \underline{0} \\ \underline{0} & \underline{M} & \cdot & \cdot & \cdot & \cdot & \underline{0} \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \underline{0} & \underline{0} & \cdot & \cdot & \cdot & \cdot & \underline{M} \end{pmatrix}}_{n\alpha} \; n\alpha \qquad (\underline{M} \text{ is } nxn)$$

$$\underline{\Phi}_{id} = \begin{pmatrix} \Phi_1(r_i) & & & & & & & \\ & \ddots & & & & & \underline{0} & \\ & & \Phi_1(r_i) & & & & & \\ & & & \Phi_1(r_i) & & & & \\ & & & & \ddots & & & \\ & & & & & \Phi_1(r_i) & & \\ & \underline{0} & & & & & \ddots & \\ & & & & & & \Phi_\alpha(r_i) & \\ & & & & & & & \ddots \\ & & & & & & & & \Phi_\alpha(r_i) \end{pmatrix}$$

$$\underline{I}_{n\alpha} = \begin{pmatrix} \underline{I} & \underline{I} & \cdots & \underline{I} \\ \underline{I} & \underline{I} & & \underline{I} \\ \vdots & & & \vdots \\ \underline{I} & \underline{I} & \cdots & \underline{I} \end{pmatrix}$$

($\underline{I}$ is the nxn identity matrix)

It is a straightforward matter to verify that $\underline{g}$ may be expressed in terms of these matrices, as:

$$\underline{g} = \sum_{i=1}^{S} \underline{A}_{id}\, \underline{M}_d\, \underline{I}_{n\alpha}\, \underline{\Phi}_{id}\, \underline{x}\,(t_{i\beta})$$

Defining $\underline{G}_i = \underline{A}_{id}\, \underline{M}_d\, \underline{I}_{n\alpha}\, \underline{\Phi}_{id}$ enables $\underline{g}$ to be written:

$$\underline{g} = \sum_{i=1}^{S} \underline{G}_i\, \underline{x}\,(t_{i\beta})$$

Appendix (continued)

2. Manipulation of difference equation (6):

$$\underline{x}[\beta S + k + 1)\tau] = e^{\underline{B}\tau}\underline{x}[(\beta S + k)\tau] + \sum_{i=1}^{S} \underline{P}_i \; \underline{x}(t_{i\beta}) \qquad (6)$$

Let $\beta S = M$

$$k = 0: \quad \underline{x}[(m+1)\tau] = e^{\underline{B}\tau}\underline{x}(M\tau) + \underline{P}_i\underline{x}\,(M\tau) + \underline{P}_S\,\underline{x}[(M-1)\tau]$$
$$+ \underline{P}_{S-1}\,\underline{x}[(M-2)\tau] + \ldots + \underline{P}_3\underline{x}[(M+2-S)\tau]$$
$$+ \underline{P}_2\underline{x}\,[(M+1-S)\tau]$$

$$k = 1: \quad \underline{x}[(M+2)\tau] = e^{\underline{B}\tau}\underline{x}[(M+1)\tau] + \underline{P}_2\,\underline{x}[(M+1)\tau] + \underline{P}_1\underline{x}(M\tau)$$
$$+ \underline{P}_S\underline{x}[(M-1)\tau] + \ldots + \underline{P}_3\underline{x}[(M+2-S)\tau]$$

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$$k = S-1: \quad \underline{x}[(M+S)\tau] = e^{\underline{B}\tau}\,\underline{x}[(M+S-1)\tau] + \underline{P}_S\underline{x}[(M+S)\tau]$$
$$+ \underline{P}_{S-1}\,\underline{x}[(M+S-2)\tau] + \ldots + \underline{P}_2\underline{x}[(M+1)\tau]$$
$$+ \underline{P}_1\,\underline{x}(M\tau)$$

If the k = 0 through k = S-2 equations are substituted into the k = S-1 equation, the following equation results:

$$\underline{x}[(M+S)\tau] = \underline{Q}_1\,\underline{x}(M\tau) + \underline{Q}_2\underline{x}[(M-1)\tau] + \underline{Q}_3\,\underline{x}[(M-2)\tau] + \ldots$$
$$+ \underline{Q}_{S-1}\,\underline{x}[(M+2-S)\tau] + \underline{Q}_S\,\underline{x}[(M+1-S)\tau]$$

where:

$$\underline{Q}_1 = \underline{P}_1 + \underline{P}_2[e^{\underline{B}\tau} + \underline{P}_1] + \underline{P}_3\left\{[e^{\underline{B}\tau} + \underline{P}_2][e^{\underline{B}\tau} + \underline{P}_1] + \underline{P}_1\right\} + \ldots$$

$$\underline{Q}_2 = [\underline{P}_2 + \underline{P}_3\left\{[e^{\underline{B}\tau} + \underline{P}_2] + I\right\} + \ldots\ldots\ldots \quad]\,\underline{P}_3$$

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$$\underline{Q}_S = [\underline{P}_2 + \underline{P}_3\;(e^{\underline{B}\tau} + \underline{P}_2) + \underline{P}_4\left\{[e^{\underline{B}\tau} + \underline{P}_3][e^{\underline{B}\tau} + \underline{P}_2] + \underline{P}_2\right\}$$
$$+ \ldots]\,\underline{P}_2$$

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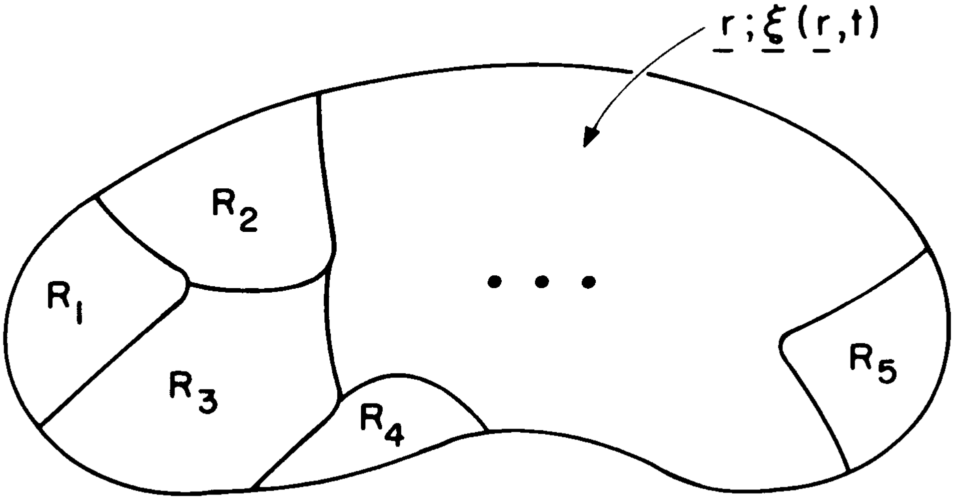
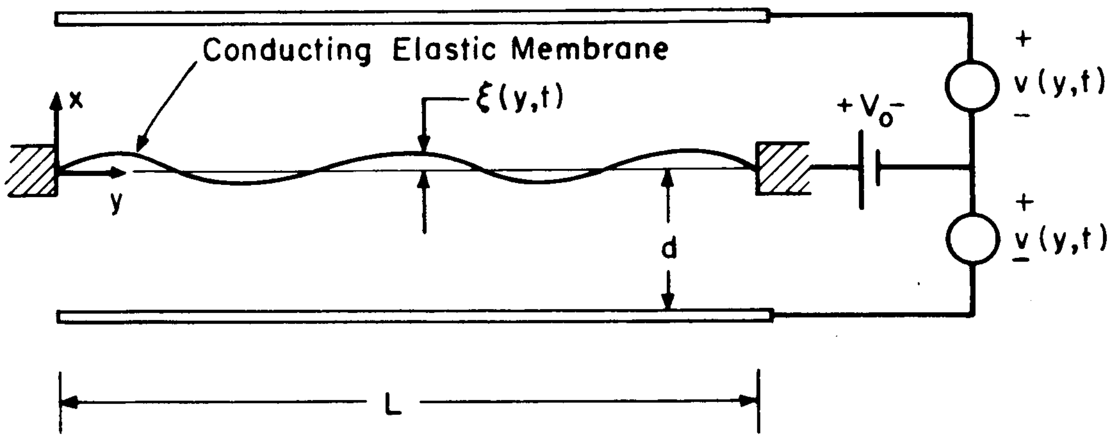
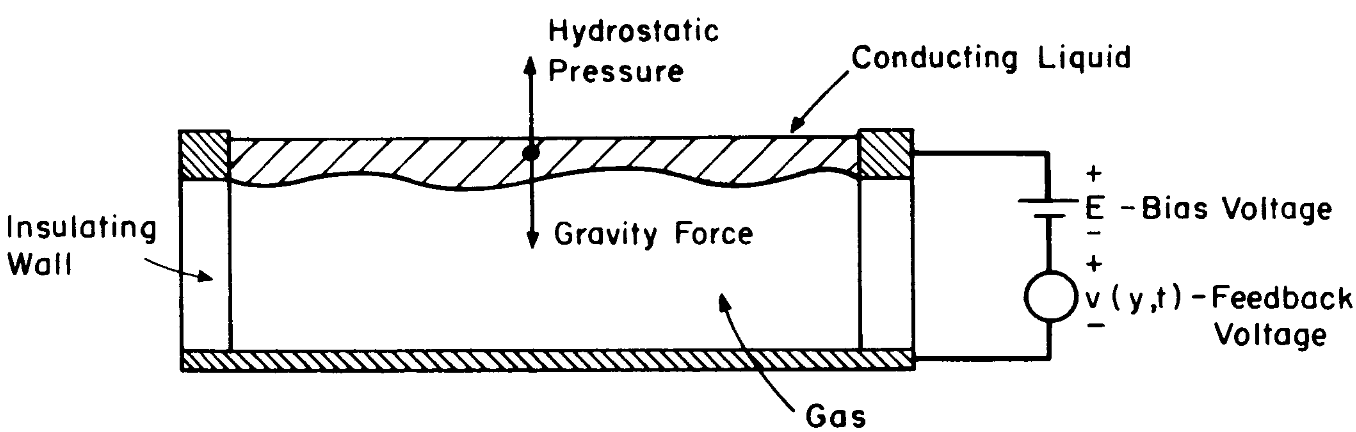


Figure 1

A continuum divided into S regions. Associated with the continuum is the variable $\underline{\xi}(\underline{r},t)$.



(a)



(b)

Figure 2

(a) The one-dimensional conducting membrane system

(b) Conducting fluid supported against gravity by hydrostatic pressure. Electric fields are used to stabilize the liquid-gas interface.

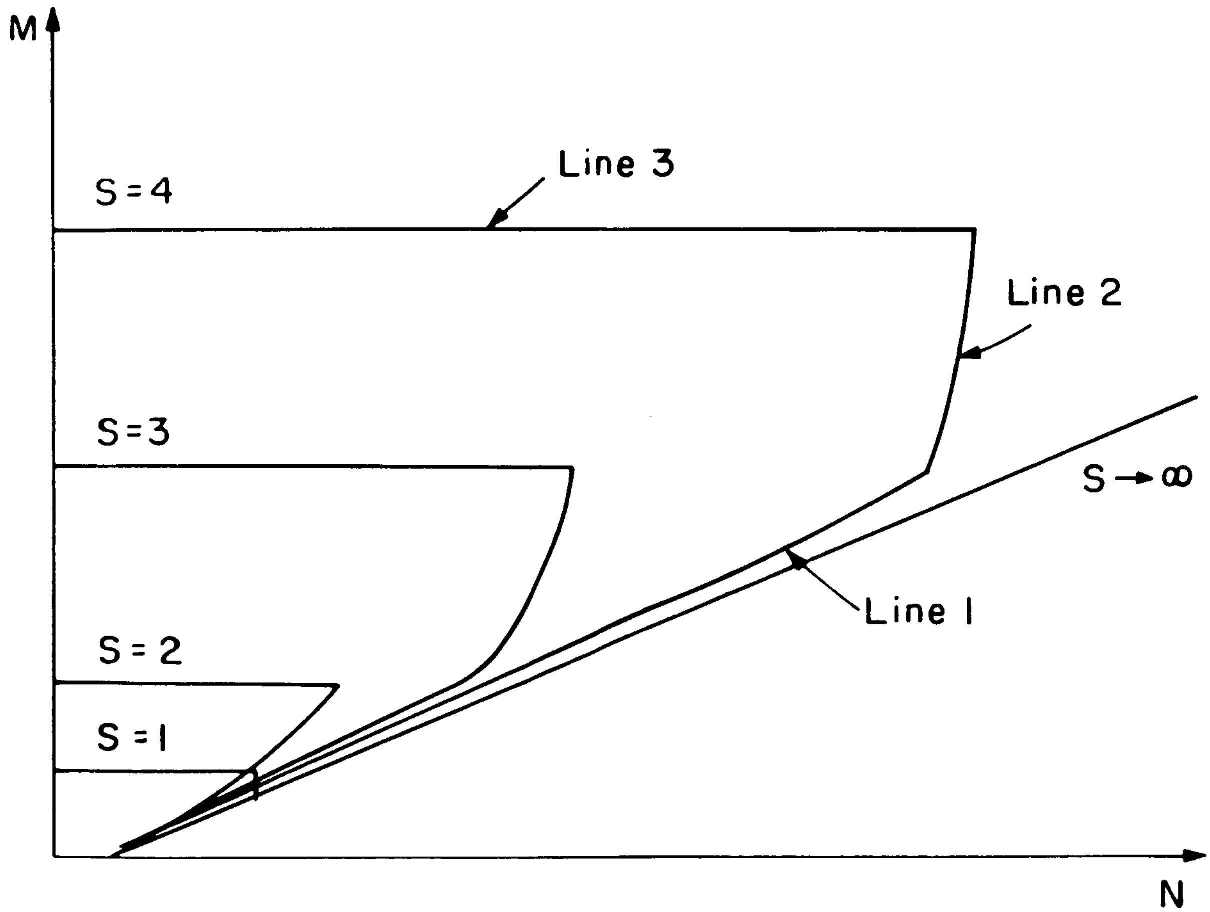


Figure 3

M-N plane plot of Melcher's[7] results for one, two, three, four, and infinite station systems.

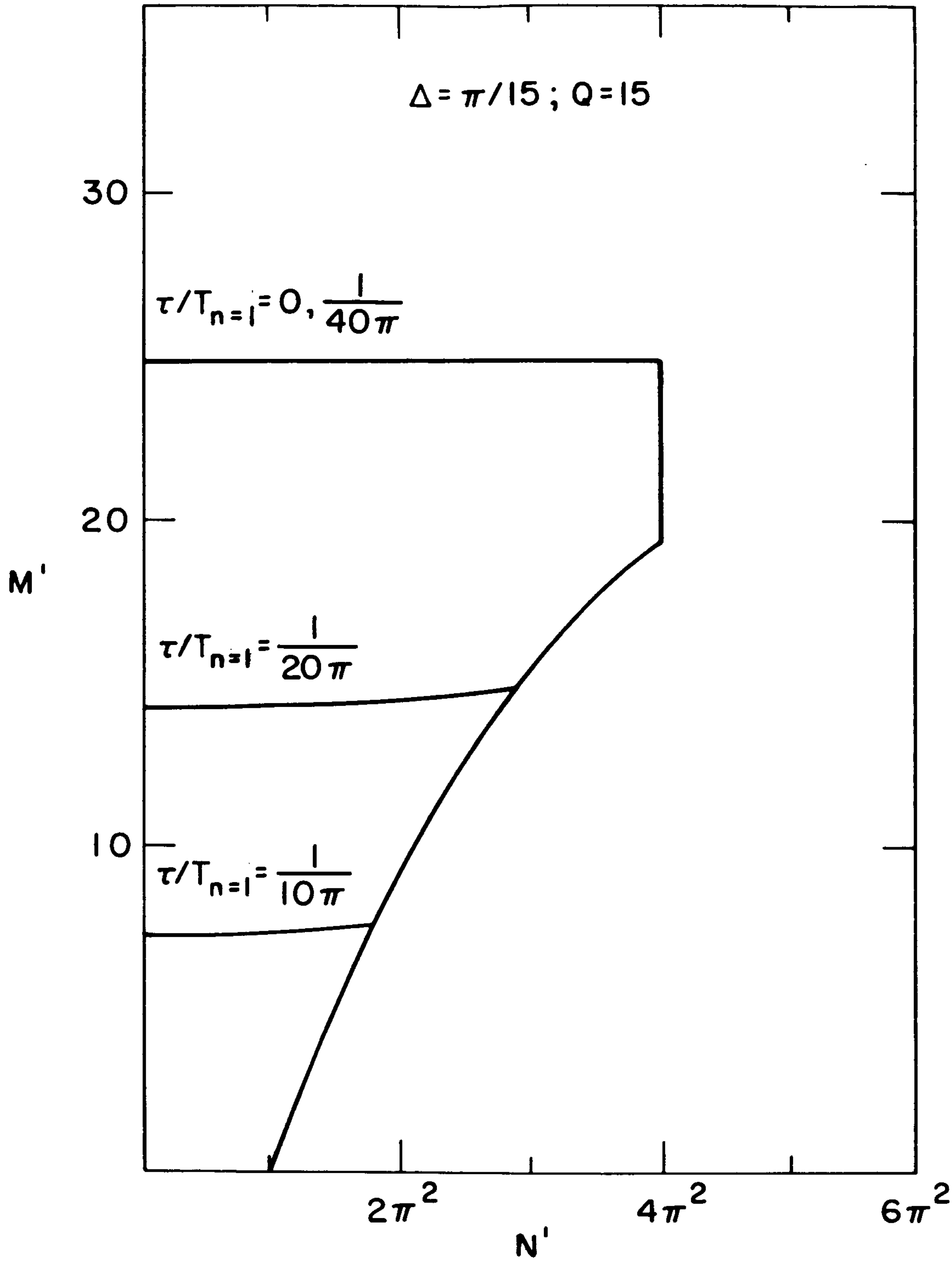


Figure 4.a

Stability characteristics of a lightly damped system

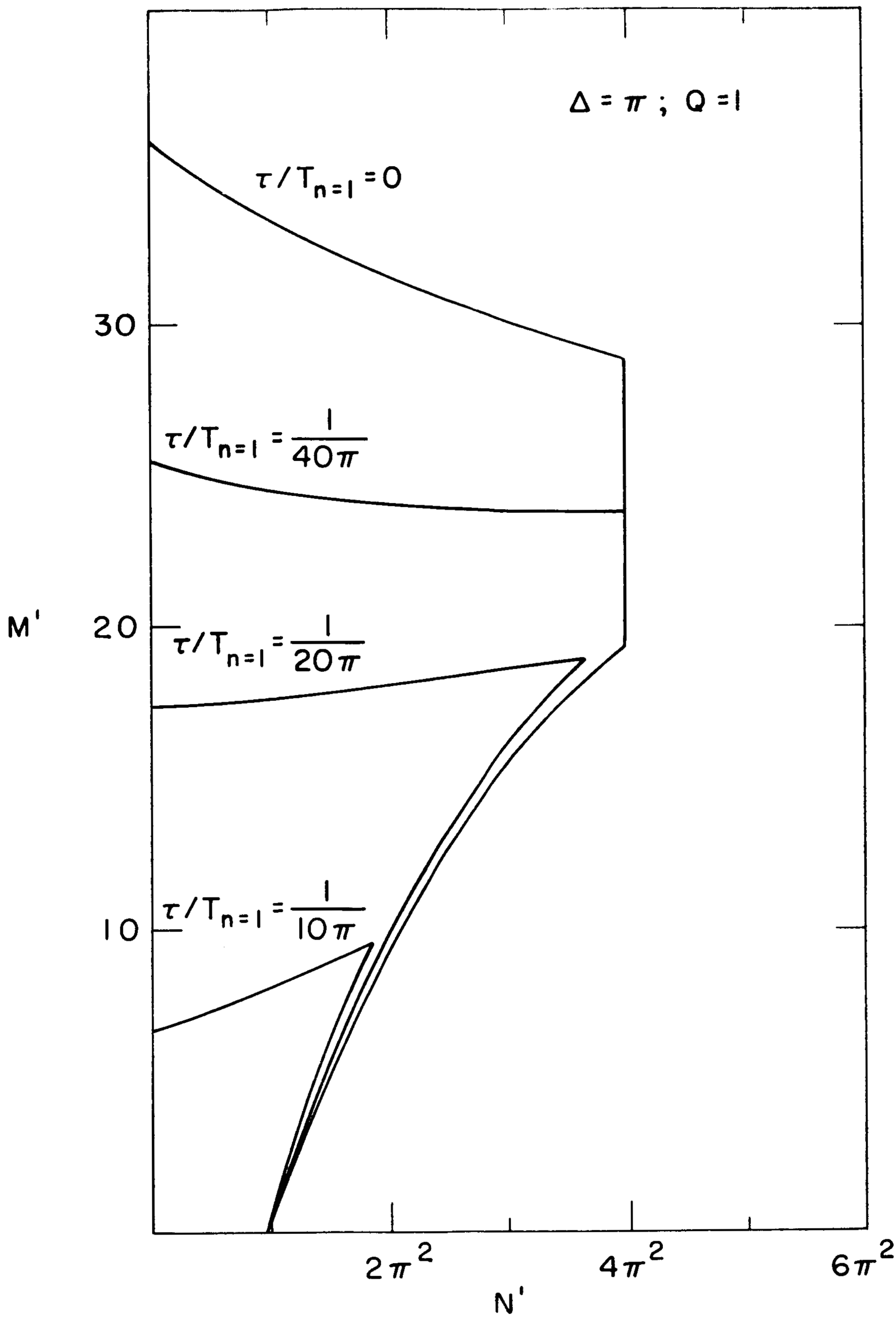


Figure 4.b

Stability characteristics of a heavily damped system